

COMMON MULTIPLES OF PATH, STAR AND CYCLE WITH COMPLETE BIPARTITE GRAPHS

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Abstract: A graph G is a common multiple of two graphs H_1 and H_2 if there exists a decomposition of G into edge-disjoint copies of H_1 and also a decomposition of G into edge-disjoint copies of H_2 . If G is a common multiple of H_1 and H_2 , and G has q edges, then we call G a (q, H_1, H_2) graph. Our paper deals with the following question: Given two graphs H_1 and H_2 , for which values of q does there exist a (q, H_1, H_2) graph? when H_1 is either a path or a star or a cycle and H_2 is a complete bipartite graph.

Keywords and Phrases: Graph Decomposition, Common Multiples of Graphs, Path, Star, Cycle, Complete Bipartite Graph.

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1. Introduction

All graphs considered here are finite and undirected unless otherwise noted. Let $|V(G)|$ and $e(G)$ denote, respectively, the order of a graph G and the size of G , that is, the number of edges in G .

K_n denotes the complete graph on n vertices, and $K_{m,n}$ denotes the complete bipartite graph with vertex partitions of cardinality m and n . A k -path, denoted

by P_k , is a path with k vertices (is a path of length $k - 1$); a k -star, denoted by S_k , is the complete bipartite graph $K_{1,k}$; a k -cycle, denoted by C_k , is a cycle of length k .

Let G and H be graphs. A *decomposition* of G is a set of edge-disjoint subgraphs of G whose union is G . An H -*decomposition* of G is a decomposition of G into copies of H . If G has an H -decomposition, we say that G is H -decomposable or H divides G and write $H|G$.

Given two graphs H_1 and H_2 , one may ask for a graph G that is a *common multiple* of H_1 and H_2 in the sense that both H_1 and H_2 divide G . Several authors have investigated the problem of finding *least common multiples* of pairs of graphs; that is, graphs of minimum size which are both H_1 - and H_2 -decomposable. The problem was introduced by Chartrand et al in [4] and they showed that every two nonempty graphs have a least common multiple. It is clear that least common multiple of two graphs may not be unique. The size of a least common multiple of two graphs H_1 and H_2 is denoted by $lcm(H_1, H_2)$. Also if q_1 and q_2 are two natural numbers, their number theoretic lcm is denoted by $lcm(q_1, q_2)$ as usual. Clearly, for two graphs H_1 and H_2 , $lcm(H_1, H_2) \geq lcm(e(H_1), e(H_2))$. The problem of finding the size of a least common multiple of graphs has been studied for several pairs of graphs: cycles and stars [4, 11], paths and complete graphs [9], pairs of cycles [8], pairs of cubes [2]. Pairs of graphs having a unique least common multiple were investigated in [5] and least common multiples of digraphs were considered in [6].

If G is a common multiple of H_1 and H_2 , and G has q edges, then we call G a (q, H_1, H_2) graph. An obvious necessary condition for the existence of a (q, H_1, H_2) graph is that $e(H_1)|q$ and $e(H_2)|q$. This obvious necessary condition is not sufficient. Some necessary conditions are easy to see and others are more difficult. For example there is no $(15, K_3, K_6)$ graph as there is no K_3 -decomposition of K_6 . However, the non-existence of a $(36, K_3, K_4)$ graph is somewhat less obvious. Hence a natural question is: *Given two graphs H_1 and H_2 , for which values of q , does there exist a (q, H_1, H_2) graph?* Adams, Bryant, and Maenhaut [1] gave a complete solution to this problem in the case where H_1 is the 4-cycle and H_2 is a complete graph; Bryant and Maenhaut [3] gave a complete solution to this problem in the case where H_1 is the complete graph K_3 and H_2 is a complete graph. A complete solution to this problem in the case where H_1 is a path and H_2 is a star is investigated in [7].

In this paper we establish necessary and sufficient condition for the existence of a $(q, P_4, K_{m,n})$ graph, a $(q, P_5, K_{m,n})$ graph, a $(q, S_3, K_{m,n})$ graph, a $(q, S_4, K_{m,n})$ graph and a $(q, C_4, K_{m,n})$ graph. The graph theoretic concepts described here are, of course, suggested by their number theoretic counterparts.

2. Preliminaries

In this section we collect some needed terminologies and notations, and present some results which are useful for our discussions. The complete graph with vertex set $\{v_1, v_2, \dots, v_m\}$ will be denoted by $[v_1, v_2, \dots, v_m]$, the m -cycle C_m with vertex set $\{v_1, v_2, \dots, v_m\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_m, v_1\}$ will be denoted by $(v_1, v_2, \dots, v_m)$, the m -path P_m with vertex set $\{v_1, v_2, \dots, v_m\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{m-1}, v_m\}$ will be denoted by $\langle v_1, v_2, \dots, v_m \rangle$ and the m -star S_m with vertex set $\{v_0, v_1, v_2, \dots, v_m\}$ and center at v_0 will be denoted by $[v_0; v_1, v_2, \dots, v_m]$. If G and H are graphs, and H is a subgraph of G , then the graph obtained by removing the edges of H from G will be denoted by $G - H$. If G_1 and G_2 are graphs, then the *union* of G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. (We shall only be considering the union of edge-disjoint graphs.)

We recall three results on P_{k+1} -decomposition, S_k -decomposition, and C_k -decomposition of $K_{m,n}$ as follows.

Theorem 1. [10] *Let k, m , and n be positive integers. There exists a P_{k+1} -decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod{k}$ and one of the cases in Table 1 occurs:*

Table 1: Necessary and Sufficient Conditions for P_{k+1} -Decomposition of $K_{m,n}$

Case	k	m	n	Characterization
1.	even	even	even	$k \leq 2m, k \leq 2n$, not both equalities
2.	even	even	odd	$k \leq 2m - 2, k \leq 2n$
3.	even	odd	even	$k \leq 2m, k \leq 2n - 2$
4.	odd	even	even	$k \leq 2m - 1, k \leq 2n - 1$
5.	odd	even	odd	$k \leq 2m - 1, k \leq n$
6.	odd	odd	even	$k \leq m, k \leq 2n - 1$
7.	odd	odd	odd	$k \leq m, k \leq n$

Theorem 2. [13] *Let k, m , and n be positive integers with $m \leq n$. There exists an S_k -decomposition of $K_{m,n}$ if and only if one of the following conditions holds:*

1. $m \geq k$ and $mn \equiv 0 \pmod{k}$;
2. $m < k \leq n$ and $n \equiv 0 \pmod{k}$.

Theorem 3. [11] *Let k, m , and n be positive integers. $K_{m,n}$ has a C_{2k} -decomposition if and only if m and n are even, $k \geq 2, m \geq k, n \geq k$, and $mn \equiv 0 \pmod{2k}$.*

We will use the following theorem on the least common multiple of two bipartite graphs by O. Favaron and C. M. Mynhardt.

Theorem 4. [8] *If F and G are bipartite, then $\text{lcm}(F, G) \leq e(F)e(G)$, where equality holds if $\gcd(e(F), e(G)) = 1$.*

3. Common Multiples of P_4 and $K_{m,n}$

In this section we determine, for all positive integers m and n , the set of integers q for which there exists a common multiple of P_4 (4 -path) and $K_{m,n}$ having precisely q edges.

Theorem 5. *There exists a graph with q edges that is both P_4 -decomposable and $K_{m,n}$ -decomposable if and only if*

1. $q \equiv 0 \pmod{3}$ and $q \equiv 0 \pmod{mn}$; and
2. $q \neq mn$ when $m = 1, n \equiv 0 \pmod{3}$ (or $n = 1, m \equiv 0 \pmod{3}$).

Proof. If there exists a $(q, P_4, K_{m,n})$ graph, then we require that 3 divides q and that mn divides q . Necessity of (1) is obvious. If $m = 1, n \equiv 0 \pmod{3}$, or $n = 1, m \equiv 0 \pmod{3}$, then $\text{lcm}(3, mn) = 3$, but $K_{m,n}$ is not P_4 -decomposable (Theorem 1). So $q \neq mn$.

To prove the sufficient conditions consider the following cases.

Case 1. $\gcd(3, mn) = 1$.

Since P_4 and $K_{m,n}$ are bipartite graphs and $\gcd(3, mn) = 1$, $\text{lcm}(P_4, K_{m,n}) = 3mn$ by Theorem 4. Therefore there exists a $(kmn, P_4, K_{m,n})$ graph for all $k \equiv 0 \pmod{3}$.

Case 2. $\gcd(3, mn) = 3$.

Then either $3|m$ or $3|n$, since 3 is a prime number. So this case can be divided into two subcases.

Case 2.1. $m, n \geq 2$.

By Theorem 1, $P_4|K_{m,n}$ for all $m, n \geq 2$ and hence there exists a $(kmn, P_4, K_{m,n})$ graph for all $k \geq 1$.

Case 2.2. $m = 1, n \equiv 0 \pmod{3}$ (or $n = 1, m \equiv 0 \pmod{3}$).

Suppose that $m = 1, n \equiv 0 \pmod{3}$. Let $n = 3r$.

Now it is sufficient to construct a $(kmn, P_4, K_{m,n})$ graph G for all $k > 1$. For this we let G be $K_{k,n}$, where $k \geq 2$. Then G can be decomposed into k edge-disjoint copies of $K_{1,n}$ and G is P_4 -decomposable by Theorem 1. Similarly we can prove the case when $n = 1, m \equiv 0 \pmod{3}$ since $K_{m,n} \cong K_{n,m}$.

4. Common Multiples of P_5 and $K_{m,n}$

In this section we determine, for all positive integers m and n , the set of integers q for which there exists a common multiple of P_5 (5-path) and $K_{m,n}$ having precisely q edges.

Theorem 6. *There exists a graph with q edges that is both P_5 -decomposable and $K_{m,n}$ -decomposable if and only if*

1. $q \equiv 0 \pmod{4}$ and $q \equiv 0 \pmod{mn}$; and
2. $q \neq mn$ when $m = n = 2$, $m = 1, n \equiv 0 \pmod{4}$ or $n = 1, m \equiv 0 \pmod{4}$.

Proof. If there exists a $(q, P_5, K_{m,n})$ graph, then we require that 4 divides q and that mn divides q . Necessity of (1) is obvious. If $m = n = 2$, then $\text{lcm}(4, mn) = 4$, but $K_{2,2}$ is not P_5 -decomposable. So $q \neq 4$. If $m = 1, n \equiv 0 \pmod{4}$, or $n = 1, m \equiv 0 \pmod{4}$, then $\text{lcm}(4, mn) = 4$, but $K_{m,n}$ is not P_5 -decomposable by Theorem 1. So $q \neq mn$.

To prove the sufficient conditions consider the following cases.

Case 1. $\gcd(4, mn) = 1$.

Then m and n are odd numbers. Since P_5 and $K_{m,n}$ are bipartite graphs and $\gcd(4, mn) = 1$, $\text{lcm}(P_5, K_{m,n}) = 4mn$ by Theorem 4. Therefore there exists a $(kmn, P_5, K_{m,n})$ graph for all $k \equiv 0 \pmod{4}$.

Case 2. $\gcd(4, mn) = 2$.

Then $m \equiv 2 \pmod{4}$ and n is odd (or $n \equiv 2 \pmod{4}$ and m is odd). Without loss of generality we may assume that $m \equiv 2 \pmod{4}$ and n is odd.

Since $\text{lcm}(4, mn) = 2mn$, it is sufficient to construct a $(2mn, P_5, K_{m,n})$ graph G , as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of this.

If $m = 2, n = 1$, to construct a $(4, P_5, K_{m,n})$ graph G , we let G be P_5 , which is both P_5 -decomposable and $K_{2,1}$ -decomposable. If $m > 2, n = 1$, to construct a $(2m, P_5, K_{m,1})$ graph G , we let G be $K_{m,2}$, which is both P_5 -decomposable (Theorem 1) and $K_{m,1}$ -decomposable. If $m \geq 2, n > 1$, to construct a $(2mn, P_5, K_{m,n})$ graph G , we let G be $K_{m,2n}$. Clearly G can be decomposed into two edge-disjoint copies of $K_{m,n}$. G is P_5 -decomposable by Theorem 1.

Case 3. $\gcd(4, mn) = 4$.

This case can be divided into four subcases.

Case 3.1. $m \equiv 0 \pmod{4}$, $n = 1$ (or $n \equiv 0 \pmod{4}$, $m = 1$).

Then $4|mn$. The graph $K_{m,n}$ is not P_5 -decomposable and there is no graph with mn edges which is both $K_{m,n}$ -decomposable and P_5 -decomposable. Now it is sufficient to construct a $(kmn, P_5, K_{m,n})$ graph G for all $k > 1$. For this case we let G

be $K_{m,k}$, $k \geq 2$. G can be decomposed into k edge-disjoint copies of $K_{m,1}$. For all $k \geq 2$, G is P_5 -decomposable by Theorem 1.

Case 3.2. $m \equiv 0 \pmod{4}$, $n > 1$, an odd number (or $n \equiv 0 \pmod{4}$, $m > 1$, an odd number).

Let $m = 4r$ and $n = 2s + 1$, where $r, s \geq 1$. Then by Theorem 1, $P_5 | K_{m,n}$.

Case 3.3. $m = n = 2$.

It is sufficient to construct an $(8, P_5, K_{2,2})$ graph and a $(12, P_5, K_{2,2})$ graph and all the required graphs can be constructed as the vertex disjoint union of appropriate number of copies of these.

To construct an $(8, P_5, K_{2,2})$ graph G , we let G be the union of the following two edge-disjoint copies of $K_{2,2}$ (i.e. C_4):

$(1, 2, 3, 4)$ and $(4, 5, 6, 7)$.

A P_5 -decomposition of G is given by the following two edge-disjoint copies of P_5 :

$\langle 2, 3, 4, 5, 6 \rangle$ and $\langle 2, 1, 4, 7, 6 \rangle$.

To construct a $(12, P_5, K_{2,2})$ graph G , we let G be the union of the following three edge-disjoint copies of $K_{2,2}$ (i.e. C_4):

$(1, 2, 3, 4)$, $(4, 5, 6, 7)$ and $(6, 8, 9, 10)$.

A P_5 -decomposition of G is given by the following three edge-disjoint copies of P_5 :

$\langle 1, 2, 3, 4, 5 \rangle$, $\langle 5, 6, 10, 9, 8 \rangle$ and $\langle 8, 6, 7, 4, 1 \rangle$.

Case 3.4 $m, n \equiv 0 \pmod{2}$, $m, n \geq 4$.

Then $4 | mn$ and by Theorem 1, $P_5 | K_{m,n}$.

Cases 3.1 and 3.3 allow us to construct the $(q, P_5, K_{m,n})$ graphs that we require for the sufficient condition(2) of Theorem 6.

5. Common Multiples of S_3 and $K_{m,n}$

In this section we determine, for all positive integers m and n , the set of integers q for which there exists a common multiple of S_3 (β -star) and $K_{m,n}$ having precisely q edges.

Theorem 7. *There exists a graph with q edges that is both S_3 -decomposable and $K_{m,n}$ -decomposable if and only if $q \equiv 0 \pmod{3}$ and $q \equiv 0 \pmod{mn}$.*

Proof. If there exists a $(q, S_3, K_{m,n})$ graph, then clearly we require 3 divides q and mn divides q .

To prove the sufficient conditions consider the following cases.

Case 1. $\gcd(3, mn) = 1$.

Since S_3 and $K_{m,n}$ are bipartite and $\gcd(3, mn) = 1$, $\text{lcm}(S_3, K_{m,n}) = 3mn$, by Theorem 4. In this case we can take a $(q, S_3, K_{m,n})$ graph G as $K_{3m,n}$ or $K_{m,3n}$.

Both graphs contain $3mn$ edges and G can be decomposed into 3 edge-disjoint copies of $K_{m,n}$ and mn edge-disjoint copies of $K_{1,3}$ (i.e. S_3). Therefore there exists a $(kmn, S_3, K_{m,n})$ graph for all $k \equiv 0 \pmod{3}$.

Case 2. $\gcd(3, mn) = 3$.

In this case either $3|m$ or $3|n$, since 3 is a prime number. Then $S_3|K_{m,n}$, by Theorem 2. So there exists a $(kmn, S_3, K_{m,n})$ graph for all $k \geq 1$.

6. Common Multiples of S_4 and $K_{m,n}$

In this section we determine, for all positive integers n , the set of integers q for which there exists a common multiple of S_4 (4 -star) and $K_{m,n}$ having precisely q edges.

Theorem 8. *There exists a graph with q edges that is both S_4 -decomposable and $K_{m,n}$ -decomposable if and only if*

1. $q \equiv 0 \pmod{4}$, $q \equiv 0 \pmod{mn}$; and
2. $q \neq 4$ when $m = n = 2$.

Proof. If there exists a $(q, S_4, K_{m,n})$ graph, then clearly we require 4 divides q and mn divides q . If $m = n = 2$, then $mn = 4$ and there is no $(4, S_4, K_{m,n})$ graphs. A graph with 4 edges containing S_4 is S_4 itself, but S_4 is not $K_{2,2}$ -decomposable. Similarly a graph with 4 edges containing $K_{2,2}$ is not S_4 -decomposable. So $q \neq 4$ when $m = n = 2$.

To prove the sufficient conditions consider the following cases.

Case 1. $\gcd(4, mn) = 1$.

Then m and n are odd numbers. Since S_4 and $K_{m,n}$ are bipartite graphs and $\gcd(4, mn) = 1$, $\text{lcm}(S_4, K_{m,n}) = 4mn$, by Theorem 4. Therefore there exists a $(kmn, S_4, K_{m,n})$ graph for all $k \equiv 0 \pmod{4}$.

Case 2. $\gcd(4, mn) = 2$.

Without loss of generality we may assume that m is an even number not divisible by 4 and n is an odd number.

Let $m = 2r$ and $n = 2s + 1$, where $r > 1, s \geq 0$.

Let $G = K_{2m,n}$ be the required $(q, S_4, K_{m,n})$ graph. Clearly G has $2mn$ edges and G can be decomposed into two edge-disjoint copies of $K_{m,n}$.

$K_{2m,n} = K_{4r,n}$ can be decomposed into rn edge-disjoint copies of $K_{4,1}$ (i.e. S_4). Thus an S_4 -decomposition of G is obtained.

Therefore there exist $(kmn, S_4, K_{m,n})$ graphs for all even k .

Case 3. $\gcd(4, mn) = 4$.

Here we need to consider three cases.

Case 3.1. $m \equiv 0 \pmod{4}$, $n \geq 1$ (or $n \equiv 0 \pmod{4}$, $m \geq 1$).

Let $m = 4r$ and $n \geq 1$. Then $K_{m,n} = K_{4r,n}$ can be decomposed into rn copies of $K_{4,1}$ (i.e. S_4). In this case $S_4 | K_{m,n}$.

Case 3.2. $m \equiv 2 \pmod{4}$ & $n \equiv 2 \pmod{4}$.

Let $m = 4r + 2$ and $n = 4s + 2$.

Then by Theorem 2, $S_4 | K_{m,n}$ for all r and s except the case when $r = s = 0$ (i.e. $m = n = 2$).

Case 3.3. $m = n = 2$.

Clearly $K_{2,2}$ is not S_4 -decomposable. For a $(8, S_4, K_{2,2})$ graph G , consider $G = K_{2,4}$ and it is S_4 -decomposable and $K_{2,2}$ -decomposable.

To construct a $(12, S_4, K_{2,2})$ graph G , we let G be the union of following three edge-disjoint copies of $K_{2,2}$ (or C_4):

$(0, 1, 2, 3)$, $(5, 4, 7, 2)$ and $(4, 6, 0, 8)$.

An S_4 -decomposition of G is given by the following three edge-disjoint copies of S_4 :

$[0; 1, 3, 6, 8]$, $[2; 1, 3, 5, 7]$ and $[4; 5, 6, 7, 8]$.

All other required graphs can be constructed by the vertex disjoint union of appropriate number of copies of $(8, S_4, K_{2,2})$ and $(12, S_4, K_{2,2})$.

7. Common Multiples of C_4 and $K_{m,n}$

In this section we determine, for all positive integers m and n , the set of integers q for which there exists a common multiple of C_4 (4 -cycle) and $K_{m,n}$ having precisely q edges.

Theorem 9. *There exists a graph with q edges that is both C_4 -decomposable and $K_{m,n}$ -decomposable if and only if*

1. $q \equiv 0 \pmod{4}$ and $q \equiv 0 \pmod{mn}$; and
2. $q \neq mn$ when $4|m$, n is an odd number (or $4|n$, m is an odd number).

Proof. If there exists a $(q, C_4, K_{m,n})$ graph, then clearly we require 4 divides q and mn divides q . Consider the cases when $4|m$ and n is an odd number, or $4|n$ and m is an odd number. We prove only one case since $K_{m,n} \cong K_{n,m}$. Suppose that $4|m$ and n is an odd number. Then $4|mn$, but by Theorem 3, $K_{m,n}$ is not C_4 -decomposable since n is odd. So $q \neq mn$ in this case.

To prove that these necessary conditions are sufficient consider the following three cases and construct the $(q, C_4, K_{m,n})$ graphs required to prove Theorem 9.

Case 1. $\gcd(4, mn) = 1$ (i.e. m and n are odd).

Since C_4 and $K_{m,n}$ are bipartite graphs and $\gcd(4, mn) = 1$, $\text{lcm}(C_4, K_{m,n}) = 4mn$ by Theorem 4. Therefore there exist $(kmn, C_4, K_{m,n})$ graphs for all $k \equiv 0 \pmod{4}$.

Case 2. $\gcd(4, mn) = 2$.

Without loss of generality we may assume that $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{2}$.
Let $m = 2r$ and $n = 2s + 1$.

Let $G = K_{m,2n}$. Clearly G has $2mn$ edges and G can be decomposed into 2 edge-disjoint copies of $K_{m,n}$.

Also $K_{m,2n} = K_{2r,2n}$ can be decomposed into rn copies of $K_{2,2}$ (i.e. C_4). Thus a C_4 -decomposition of G obtained.

Therefore there exist $(kmn, C_4, K_{m,n})$ graphs for all even k .

Case 3. $\gcd(4, mn) = 4$.

We need to consider two cases.

Case 3.1. $m \equiv 0 \pmod{4}$ and n is an odd number (or m is an odd number and $n \equiv 0 \pmod{4}$).

Let $m = 4r$ and n be an odd number. Clearly $K_{m,n}$ is not C_4 -decomposable since it contains odd degree vertices.

We have to show that there exists a $(kmn, C_4, K_{m,n})$ graph for all $k \geq 2$.

It is sufficient to construct a $(8rn, C_4, K_{m,n})$ graph and a $(12rn, C_4, K_{m,n})$ graph as all the required graphs can be constructed as the vertex-disjoint union of an appropriate number of copies of $(8rn, C_4, K_{m,n})$ graphs and $(12rn, C_4, K_{m,n})$ graphs.

To construct a $(8rn, C_4, K_{m,n})$ graph G , we let $G = K_{4r,2n}$.

Then G can be decomposed into 2 copies of $K_{m,n}$ and $2rn$ edge-disjoint copies of $K_{2,2}$ (i.e. C_4).

To construct a $(12rn, C_4, K_{m,n})$ graph G , let $G_1 = K_{4r,n}$ with vertex set $V(G_1) = U \cup V$, where $U = \{u_i | i = 1, 2, \dots, 4r\}$ and $V = \{v_j | j = 1, 2, \dots, n\}$,

$E(G_1) = \{(u_i, v_j) | i = 1, 2, \dots, 4r \text{ and } j = 1, 2, \dots, n\}$.

Let $V' = \{v'_j | j = 1, 2, \dots, 2n\}$ and $U' = \{u'_i | i = 1, 2, \dots, 2r\}$.

A $(12rn, C_4, K_{m,n})$ graph G is constructed with vertex set $V(G) = U \cup V \cup V' \cup U'$ and edge set

$$\begin{aligned} E(G) = & E(G_1) \cup \{(u_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, n\} \\ & \cup \{(u_i, v'_j) | i = 2r + 1, 2r + 2, \dots, 4r \text{ and } j = n + 1, n + 2, \dots, 2n\} \\ & \cup \{(u'_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, 2n\}. \end{aligned}$$

Clearly G has $12rn$ edges.

A $K_{m,n}$ -decomposition of G into the three edge-disjoint copies of $K_{m,n}$ is given by $G_i, i = 1, 2, 3$, where G_1 is the graph defined above and G_2 is the graph with edge set

$$\begin{aligned} E(G_2) = & \{(u_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, n\} \\ & \cup \{(u'_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, n\}. \end{aligned}$$

Let G_3 be the graph with edge set

$$E(G_3) = \{(u_i, v'_j) | i = 2r + 1, 2r + 2, \dots, 4r \text{ and } j = n + 1, n + 2, \dots, 2n\} \\ \cup \{(u'_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = n + 1, n + 2, \dots, 2n\}.$$

A C_4 - decomposition of G is given by the following $3rn$ edge-disjoint copies of C_4 .
 $\{(u'_i, v'_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, 2n\}$ gives rn copies of C_4 ,
 $\{(u_i, v'_j), (u_i, v_j) | i = 1, 2, \dots, 2r \text{ and } j = 1, 2, \dots, n\}$ gives rn copies of C_4 ,
 $\{(u_i, v'_k), (u_i, v_j) | i = 2r + 1, 2r + 2, \dots, 4r, k = n + 1, n + 2, \dots, 2n \text{ and } j = 1, 2, \dots, n\}$ gives rn copies of C_4 .

Case 3.2. $m, n \equiv 0 \pmod{2}$.

Then by Theorem 3, $C_4 | K_{m,n}$ and hence there exist $(kmn, C_4, K_{m,n})$ graphs for all $k \geq 1$.

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